

‘Superresonance’ from a rotating acoustic black hole

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Using the analogy between a shrinking fluid vortex (‘draining bathtub’), modelled as a (2+1) dimensional fluid flow with a sink at the origin, and a rotating (2+1) dimensional black hole with an ergosphere, it is shown that a scalar sound wave is reflected from such a vortex with an *amplification* for a specific range of frequencies of the incident wave, depending on the angular velocity of rotation of the vortex. We discuss the possibility of observation of this phenomenon, especially for inviscid fluids like liquid HeII, where vortices with quantized angular momentum may occur.

The possibility of experimental observation of the acoustic analog of Hawking radiation from regions of flow of inviscid and barotropic fluids behaving as *outer trapped* surfaces (‘acoustic event horizons’), was pointed out by Unruh [1] more than two decades ago. Recent advances in theoretical understanding of such acoustic analogs of black holes, as well as improved experimental techniques involving Bose-Einstein condensates [2] have brought such experimental possibilities within the realm of reality. A variety of fluid flow configurations have been identified, all of which can now be described in terms of different analog black hole models [3]. Many of these can actually be realized in the laboratory under cryogenic conditions; thus, the analogy enables laboratory tests of a class of phenomena associated with those aspects of black hole physics that are insensitive to whether or not the metric satisfies the Einstein equation.

While attention so far has been focussed on the observation of Hawking radiation from these acoustic black holes, the question of possible energy extraction from these analogs via the Penrose process [4] also needs to be addressed. This process requires black holes with angular momentum. The analogous situation would correspond to fluid flow with rotation, possessing a region of supersonic flow analogous to the *ergosphere* of a Kerr black hole, surrounding the horizon. In the case of the Kerr black hole, a scalar wave (a solution of the massless Klein-Gordon equation in the black hole spacetime), entering the ergosphere, is reflected back outside with an amplitude that exceeds the amplitude of the incident wave, for a certain range of frequencies bounded from above by the angular velocity of the black hole. The wave extracts energy from the rotational energy of the hole, leading to a ‘spin-down’ of the latter. This wave-analog of the Penrose process was discovered by Zeldovich [5] and investigated in detail by Starobinsky [6] and Misner (who dubbed the phenomenon ‘superradiance’) [7].

In this paper, we investigate the possibility of the acoustic analog of superradiance (a phenomenon that we call ‘superresonance’), i.e., the amplification of a sound wave by reflection from the ‘ergo’-region of a rotating acoustic black hole. For the latter, we choose the so-called ‘draining bathtub’ type of fluid flow [3], which is basically a 2+1 dimensional flow with a sink at the origin. A two surface in this flow, on which the fluid velocity is everywhere pointing towards the sink, and the radial velocity component exceeds the local sound velocity everywhere, behaves as an outer trapped surface in this ‘acoustic’ spacetime, and is identified with the (future) event horizon of the black hole analog. Thus, the velocity potential for the flow has the form [3] (in polar coordinates on the plane)

$$\psi(r, \phi) = A \log r + B \phi, \quad (1)$$

where, A and B are real constants. This leads to the velocity profile

$$\mathbf{v} = \frac{A}{r} \hat{r} + \frac{B}{r} \hat{\phi}. \quad (2)$$

It has been shown [3] that, for barotropic and inviscid fluids with flows that are free of turbulence, a linear acoustic disturbance is described by a velocity potential that satisfies the massless general-coordinate-invariant Klein-Gordon equation,

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu) \psi = 0, \quad (3)$$

where, $g_{\mu\nu}$ is a metric (with Lorentzian signature), not of spacetime itself, but of an acoustic ‘analog spacetime’. Such analog spacetimes often resemble black hole spacetimes. For the velocity potential given in (1), the analog black hole metric is 2+1 dimensional with Lorentzian signature, and is given by

$$ds^2 = - \left(c^2 - \frac{A^2 + B^2}{r^2} \right) dt^2 - \frac{2A}{r} dr dt - 2B d\phi dt + dr^2 + r^2 d\phi^2, \quad (4)$$

where c is the velocity of sound.

The properties of this metric become clearer in slightly different coordinates, defined through the transformations of the time and the azimuthal angle coordinates in the *exterior* region ($|A|/c + 0 \leq r < \infty$),

$$\begin{aligned} dt &\rightarrow dt + \frac{|A| r}{(r^2 c^2 - A^2)} dr \\ d\phi &\rightarrow d\phi + \frac{B |A|}{(r^2 c^2 - A^2)} dr . \end{aligned} \quad (5)$$

In the new coordinates, the metric in the exterior region assumes the form, after a rescaling of the ‘time’ coordinate by c ,

$$ds^2 = - \left(1 - \frac{A^2 + B^2}{c^2 r^2} \right) dt^2 + \left(1 - \frac{A^2}{c^2 r^2} \right)^{-1} dr^2 - 2 \frac{B}{c} d\phi dt + r^2 d\phi^2 . \quad (6)$$

As for the Kerr black hole in general relativity, the radius of the ergosphere is given by the vanishing of g_{00} , i.e., $r_e = (A^2 + B^2)^{1/2}/c$. The metric has a (coordinate) singularity at $r_h = |A|/c$, which signifies the horizon, i.e., the boundary of the outer trapped surface.¹ The metric is a stationary and axisymmetric one with two Killing vector fields. Note that the fall-off of the metric components at $r \rightarrow \infty$ is faster than that of standard asymptotically flat black hole spacetime. This has the consequence that the standard definitions of globally conserved quantities in terms of Komar integrals [4] do not seem to work without modification.

We now turn to the Klein-Gordon equation (3) in the background metric (6). Given the stationarity and axisymmetry of the metric, the equation can be separated as

$$\psi(t, r, \phi) = \exp i(\omega t - m\phi) R(r) , \quad (7)$$

where, m is a real constant that is *not* restricted to assume only a discrete set of values, because we are working with only two space dimensions. This should be contrasted with the usual 3+1 dimensional treatment of superradiance [7]. We assume further that $\omega > 0$. The radial function $R(r)$ satisfies the linear second order differential equation

$$\begin{aligned} \frac{1}{r} \left(1 - \frac{A^2}{c^2 r^2} \right) \frac{d}{dr} \left[r \left(1 - \frac{A^2}{c^2 r^2} \right) \frac{d}{dr} \right] R(r) \\ + \left[\omega^2 - \frac{2Bm\omega}{cr^2} - \frac{m^2}{r^2} \left(1 - \frac{A^2 + B^2}{c^2 r^2} \right) \right] R(r) = 0 . \end{aligned} \quad (8)$$

Thus, the problem has reduced to a one dimensional Schrödinger problem.

Two further analytical simplifications can be made: first, we introduce the tortoise coordinate r^* through the equation

$$\frac{d}{dr^*} = \left(1 - \frac{A^2}{r^2 c^2} \right) \frac{d}{dr} , \quad (9)$$

which implies that

$$r^* = r + \frac{|A|}{2c} \log \left| \frac{r - \frac{|A|}{c}}{r + \frac{|A|}{c}} \right| . \quad (10)$$

Observe that the tortoise coordinate spans the entire real line as opposed to r which spans only the half-line; the horizon $r = |A|/c$ maps to $r^* \rightarrow -\infty$, while $r \rightarrow \infty$ corresponds to $r^* \rightarrow +\infty$. Next, introducing a new radial function $G(r^*) \equiv r^{1/2} R(r)$, one obtains the modified radial equation,

$$\frac{d^2 G(r^*)}{dr^{*2}} + \left[Q(r) + \frac{1}{4r^2} \left(\frac{dr}{dr^*} \right)^2 - \frac{A^2}{r^4 c^2} \left(\frac{dr}{dr^*} \right) \right] G(r^*) = 0 , \quad (11)$$

where,

¹Note that the constant A must be chosen to be negative to obtain the horizon as indicated.

$$Q(r) = \frac{A^2 m^2 + B^2 m^2 - c^2 m^2 r^2 - 2 B m r^2 \omega + r^4 \omega^2}{r^4} . \quad (12)$$

The main advantage of the new radial equation (11) over (8) is the absence in the former of a first derivative of the radial function; this implies that the Wronskian of linearly independent solutions is a constant independent of r^* (or r). This property is of crucial importance in what follows.

We analyze (11) in two distinct radial regions, viz., near the horizon, i.e., $r^* \rightarrow -\infty$ and at asymptopia, i.e., $r^* \rightarrow +\infty$. In the asymptotic region, equation (11) can be written approximately as,

$$\frac{d^2 G(r^*)}{dr^{*2}} + \omega^2 G(r^*) = 0 ; \quad (13)$$

this can be solved trivially,

$$G(r^*) = \exp(i \omega r^*) + \mathcal{R} \exp(-i \omega r^*) \equiv G_A(r^*) . \quad (14)$$

The first term in equation (14) corresponds to ingoing wave and the second term to the reflected wave, so that \mathcal{R} is the reflection coefficient in the sense of potential scattering. It is not difficult to calculate the Wronskian of the solutions (14); one obtains

$$\mathcal{W}(+\infty) = -2 i \omega (1 - |\mathcal{R}|^2) . \quad (15)$$

Near the horizon ($r^* \rightarrow -\infty$), eqn. (11) can be written approximately as,

$$\frac{d^2 G(r^*)}{dr^{*2}} + (\omega - m \Omega_H)^2 G(r^*) = 0 . \quad (16)$$

where, $\Omega_H \equiv Bc/A^2$ is identified with the angular velocity of the acoustic black hole. We impose the boundary condition that of the two solutions of this equation, only the *ingoing* one is physical, so that one has

$$G(r^*) = \mathcal{T} \exp i(\omega - m \Omega_H) r^* \equiv G_H(r^*) . \quad (17)$$

The undetermined coefficient \mathcal{T} is the transmission coefficient of the one dimensional Schrödinger problem. Once again, it is easy to calculate the Wronskian of this solution; one obtains

$$\mathcal{W}(-\infty) = -2 i (\omega - m \Omega_H) |\mathcal{T}|^2 . \quad (18)$$

Since both equations are actually limiting approximations of eq. (11), which, as we have mentioned, has a constant Wronskian, it follows that

$$\mathcal{W}(+\infty) = \mathcal{W}(-\infty) , \quad (19)$$

so that, from eqn.s (15) and (18), we obtain the relation

$$1 - |\mathcal{R}|^2 = \left(\frac{\omega - m \Omega_H}{\omega} \right) |\mathcal{T}|^2 . \quad (20)$$

It is obvious from eq. (20) that, for frequencies in the range $0 < \omega < m \Omega_H$, the reflection coefficient has a magnitude *larger than unity*. This is precisely the amplification relation that emerges in superradiance from rotating black holes in general relativity [6], [7].

For a given frequency ω in the above range and a given value of the azimuthal mode number m , the total energy flux at asymptopia can be easily calculated [8] in terms of the reflection coefficient \mathcal{R} . By conservation of energy, this flux must equal the rate of loss of mass (energy) from the black hole, so that one may write

$$\begin{aligned} \frac{dM}{dt} &= -\mathcal{P}_{\omega m}^\infty = \frac{\pi \omega^2}{c} (1 - |\mathcal{R}_{\omega m}|^2) \\ &= \frac{\pi \omega}{c} (\omega - m \Omega_H) |\mathcal{T}_{\omega m}|^2 , \end{aligned} \quad (21)$$

where, M is the mass of the acoustic black hole (which must be related to the energy of the vortex under consideration) and $\mathcal{P}_{\omega m}^\infty$ is the average power radiated to asymptopia for a given ω and m (modulo an overall positive constant proportional to the equilibrium density of the fluid). We have used eq. (20) in arriving at (21). The loss of mass

of the black hole superresonance is obvious in the above equation, so long as the frequency lies in the range quoted above. A similar equation can be derived for the time rate of change of angular momentum of the hole [7].

In order to link up more precisely with experimental possibilities, one needs to be able to compute the reflection coefficient by explicitly solving eq. (11). We shall not attempt this exercise in this paper, but postpone that discussion to a future publication [8]. Here, we shall concentrate instead on certain qualitative features of the sort of observations one might hope to make on this phenomenon. In particular, our assumption at the outset of an irrotational, nonviscous fluid flow leads us immediately to a kind of fluid that is known for decades to have truly remarkable properties, liquid HeII, or more precisely, superfluid He4. We are especially interested in the existence of vortices with quantized angular momenta. We follow [9] to obtain a rough picture of what this will do to superresonant amplitudes.

The velocity profile for fluids under consideration is given by eq. (2) and follows from (1) using standard relations of fluid dynamics. Now, let us imagine that the fluid is actually superfluid HeII and our black hole is a vortex in the fluid with a sink at the centre. In the quantum theory of HeII, the wave function is of the form

$$\Psi = \exp i \sum_i \phi(\vec{r}_i) \Phi_{ground} , \quad (22)$$

where, \vec{r}_i is the location of the i th particle of HeII. The velocity at any point is given by the gradient of the phase at that point

$$\mathbf{v} \equiv \nabla \phi . \quad (23)$$

This means that the velocity potential can be thought of crudely as the phase of the wave function. However, this phase cannot be non-singular everywhere, since we are assuming a sink at $r = 0$. Thus, there exist non-trivial holonomies of the velocity field on closed curves around this sink. For simplicity, let us concentrate on circles around the sink. The change of phase of the wave function around any of these circles, for the velocity profile given in (2) is given by

$$\Delta\phi \propto 2\pi B , \quad (24)$$

where, the constant of proportionality depends on the microscopic properties of the fluid, and is unaffected by our choice of (2). Thus, for the overall wave function to be single valued, the parameter B in the acoustic black hole metric must be proportional to an integer n .

The immediate implication of this property is that the angular velocity Ω_H at the horizon is likewise restricted to be proportional to the integer n . The likely fallout of this is ‘quantization’ of the outgoing energy flux, through the last part of eq. (21). In other words, it is conceivable that the angular momentum of the black hole will now change discontinuously from being proportional to n to being proportional to $n - 1$ due to superresonance; this will cause the energy flux also to change in discrete steps - an effect that should make observations easier than in actual black holes. This will also be manifest in the frequency spectrum of the reflection coefficient \mathcal{R} [10] : one should see a multiplicity of equally-spaced peaks in this spectrum, perhaps with differing strengths and, of course, with widths that are multiples of the smallest width. We hope to report in more quantitative terms on these interesting aspects of superresonance in a forthcoming publication.

We end the letter with the rather curious observation that at the time of the original discovery of superradiance, the phenomenon of sound amplification due to reflection from a medium at rest, with a supersonically moving boundary, was already known for about four decades [5]. It is not completely clear to us whether this is the same as superresonance. There may also be some connection of superresonance with stimulated vortex sound [11]. If indeed these classical acoustic phenomena are examples of superresonance, then one would be led to conclude that evidence already exists of acoustic black hole analogs existing in nature. On the other hand, it also means that these acoustic phenomena have a novel interpretation in terms of semiclassical black hole analog models. However, the intriguing aspects of superresonance in HeII may still be novel enough to do further experiments, to confirm all aspects of the black hole analog picture.

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Note added After submission of this paper to the Archives, we became aware of some general discussion of the possibility of superresonance in superfluid HeII [12]; however, the aspects of the phenomenon dealt with in that reference seem to be different from our concerns in this paper. We thank G. Volovik for bringing this reference to our attention. We also thank U. Fischer for correspondence regarding transformations (5) and for bringing [13] to our

attention. Reservations expressed about the validity of these transformations do not appear to be significant in the region *exterior* to the event horizon, which is our region of interest.

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